Rapid grounding line migration induced by internal ice stream variability: Supplementary material

Alexander Robel*
Christian Schoof†
Eli Tziperman*

1 Model Details

1.1 Stretch Coordinates

One of the primary purposes of this model is to accurately capture the grounding line position as the ice stream evolves. Our approach here is based off of that of Schoof (2007). We adopt a system of dimensionless stretch coordinates in the $x - z$ plane

$$\sigma = \frac{x}{x_g} \tag{1}$$

$$\eta = \frac{z - b}{h}, \tag{2}$$

where x_g is the grounding line position, b is the bedrock elevation and h is the ice stream thickness. In this system, the grounding line is always at $\sigma = 1$ and the ice surface is always at $\eta = 1$.

1.2 Horizontal Velocity

We start by considering the x-directed momentum balance for an ice stream

$$\partial_x \left(2h\bar{A}^{-\frac{1}{n}} \left| \partial_x u_b \right|^{\frac{1}{n}-1} \partial_x u_b \right) = \tau_d(x,t) + \tau_b(x,t) + G_s h |u_b|^{\frac{1}{n}-1} u_b, \tag{3}$$

where $u_b(x,t) = u(z = b; x, t)$ is the basal ice velocity. The term on the LHS is the longitudinal stress and the three terms on the RHS are (respectively) the driving stress, basal shear stress and cross-stream integrated lateral shear stress. The basal velocity u_b is assumed to result from till deformation. $\bar{A}$ is the vertically integrated Glen's law coefficient which is a function of ice temperature and n is the Glen's law exponent.

Driving stress has its usual form

$$\tau_d(x,t) = \rho_i g h \partial_x h, \tag{4}$$

*Department of Earth & Planetary Sciences, Harvard University
†Department of Earth & Ocean Sciences, University of British Columbia

where ρ_i is the density of glacial ice and g is the acceleration due to gravity.

As in Dupont and Alley (2005), the parameter G_s arises from assuming that cross-stream variation in velocity primarily occurs in the shear margins and then scaling away the parameters that arise in this margin, yielding

$$G_s = W^{-1}(A_s W_s)^{-\frac{1}{n}} \tag{5}$$

where A_s is the Nye-Glen Law coefficient in shear margins, W is the ice stream half-width and W_s is the shear margin width.

Vertical shear of horizontal velocity is calculated separately by following the shallow ice approximation and assuming simple shear

$$u(\eta) = u_b + \frac{2\bar{A}}{n+1}\tau_d^n H \left[1 - (1-\eta)^{n+1}\right]. \tag{6}$$

where η is the scaled vertical coordinate, u_b is the basal velocity calculated using the momentum balance from above and $\bar{A}$ is the vertically averaged Glen's law coefficient.

1.3 Vertical Velocity

We expect that our model may develop large gradients in horizontal velocity as horizontal variations in till water content lead to large variations in bed strength (see section 1.6). Without enforcing mass continuity, these local divergences will lead to significant losses of heat from the ice in locations where the heat balance is critical. Thus, we enforce x-z mass continuity

$$\partial_x u + \partial_z w = 0. \tag{7}$$

Adopting stretch coordinates, this becomes

$$\frac{\partial_\sigma u}{x_g} - \frac{(\eta\partial_\sigma h + \partial_\sigma b)\,\partial_\eta u}{h x_g} + \frac{\partial_\eta w}{h} = 0, \tag{8}$$

subject to $w + u\partial_x b = 0$ at $z = b$, now $w = \frac{u}{x_g}\partial_\sigma b$ at $\eta = 0$. For constant σ, we choose to integrate from the bed upwards to solve for $w(\eta)$ as a function of $u(\eta)$ (pre-computed above)

$$w(x,z,t) = \int_0^1 \left[\frac{(\eta\partial_\sigma h + \partial_\sigma b)\,\partial_\eta u}{h x_g} - \frac{\partial_\sigma u}{x_g}\right]_{\sigma=\sigma_0} d\eta. \tag{9}$$

1.4 Ice Thickness

Ice thickness evolution is the result of a simple mass balance with constant accumulation as a source everywhere and an advective flux moving ice within and out of the domain. The resulting prognostic equation for ice thickness is simply

$$\dot{h} + \partial_x(\bar{u}h) = a_c \tag{10}$$

where $\bar{u}(x,t) = \frac{1}{h}\int_b^{b+h} u(x,z,t)dz$ is the vertically averaged velocity.

Again, adopting stretch coordinates, the advection equation for ice thickness will change to

$$\dot{h} - \frac{\sigma\dot{x}_g}{x_g}\partial_\sigma h + \frac{\partial_\sigma(\bar{u}h)}{x_g} = a \tag{11}$$

where the extra terms are the result of the changing coordinate system.

We can rewrite this in a more natural divergence form

$$\dot{h} + \frac{1}{x_g}\partial_\sigma \left[\left(\bar{u} - \sigma \dot{x}_g \right) h \right] + \frac{\dot{x}_g}{x_g} h = a \tag{12}$$

where $\bar{u} - \sigma \dot{x}_g$ is the "effective" horizontal velocity less the rate of coordinate stretching.

1.5 Ice Temperature

Calculating temperature along the flowline and in the vertical is necessary in order to accurately determine the basal heat budget and attendant meltwater production rate. We begin with the advection-diffusion equation for temperature in the x-z plane

$$\dot{T} + \nabla \cdot (\vec{u}T) = \kappa \nabla^2 T \tag{13}$$

$$\dot{T} + \partial_x (uT) + \partial_z (wT) = \kappa (\partial_{xx} T + \partial_{zz} T) \tag{14}$$

We now transform to the same stretch coordinates as above. In a full expansion, we are left with the following unwieldy heat equation

$$\dot{T} - \frac{\sigma \dot{x}_g}{x_g}\partial_\sigma T - \frac{\eta}{h}\left(\dot{h} - \frac{\sigma \dot{x}_g}{x_g}\partial_\sigma h \right) \partial_\eta T + \frac{1}{x_g}\partial_\sigma (uT) - \frac{1}{h x_g}\left(\eta \partial_\sigma h + \partial_\sigma b \right) \partial_\eta (uT) + \frac{1}{h}\partial_\eta (wT) =$$

$$\kappa \left[\frac{1}{x_g^2}\partial_{\sigma\sigma} T + \frac{\eta^2 \left(\partial_\sigma h \right)^2}{h^2 x_g^2}\partial_{\eta\eta} T - \frac{2\eta \partial_\sigma h}{h x_g^2}\partial_{\eta\sigma} T + \frac{2\eta \partial_\sigma h}{h^2 x_g}\partial_\eta T - \frac{\eta \partial_{\sigma\sigma} h}{h x_g^2}\partial_\eta T + \frac{1}{h^2}\partial_{\eta\eta} T \right] \tag{15}$$

Then, within the brackets on the RHS, we can perform some scaling, assuming that: $\sigma \sim O(1)$, $\eta \sim O(1)$, $x_g \sim O(10^5)$, $h \sim O(10^3)$. This leaves only the vertical diffusion term, and so our heat equation reduces to

$$\dot{T} - \frac{\sigma \dot{x}_g}{x_g}\partial_\sigma T - \frac{\eta}{h}\left(\dot{h} - \frac{\sigma \dot{x}_g}{x_g}\partial_\sigma h \right) \partial_\eta T + \frac{1}{x_g}\partial_\sigma (uT) - \frac{1}{h x_g}\left(\eta \partial_\sigma h + \partial_\sigma b \right) \partial_\eta (uT) + \frac{1}{h}\partial_\eta (wT) = \frac{\kappa}{h^2}\partial_{\eta\eta} T \tag{16}$$

In divergence form this becomes

$$\frac{\partial T}{\partial \tau} + \frac{1}{x_g}\frac{\partial}{\partial \sigma}\left[(u - \sigma \dot{x}_g) T \right] + \frac{1}{h}\frac{\partial}{\partial \eta}\left[\left(w - \eta \left\{ \frac{\partial h}{\partial \tau} + \frac{(u - \sigma \dot{x}_g)}{x_g}\frac{\partial h}{\partial \sigma} \right\} - \frac{u}{x_g}\frac{\partial b}{\partial \sigma} \right) T \right]$$

$$+ \frac{\dot{x}_g}{x_g} T + \frac{1}{h}\left[\frac{\partial h}{\partial \tau} + \frac{(u - \sigma \dot{x}_g)}{x_g}\frac{\partial h}{\partial \sigma} \right] T = \frac{k}{h^2} T_{\eta\eta}. \tag{17}$$

The advective velocity in the η-direction turns out to be

$$w - \eta \left\{ \frac{\partial h}{\partial \tau} + \frac{(u - \sigma \dot{x}_g)}{x_g}\frac{\partial h}{\partial \sigma} \right\} - \frac{u}{x_g}\frac{\partial b}{\partial \sigma}. \tag{18}$$

This accounts for the tilting of the element boundaries through the last term as well as for the motion of the element as the domain is stretched.

1.6 Till Properties

Since till water content is determined solely by local meltwater production, it is dealt with in a similar way as in Robel et al. (2013), which adopts a slightly modified form of the undrained plastic bed model of Tulaczyk et al. (2000b). The basal heat budget is

$$m = \frac{1}{\rho_i L_f}\left(G + \tau_b(x)u_b(x) + \frac{k_i}{h(x)}\partial_\eta T(x)|_{\eta=0}\right) \tag{19}$$

where, on the RHS, the first term is the geothermal heat flux, the second term is the frictional heat flux and the third term is the conductive heat flux at the bed.

Till void ratio, $e = \frac{Z_w}{Z_s}$, is a ratio of the thickness of void spaces in the till column (Z_w) to unfrozen solid till thickness without void spaces (Z_s). Assuming that meltwater always fills the void spaces in the till column, the till water content can then be defined as $Z_w = eZ_s$. e and Z_s then vary as a function of the ice stream state.

Void ratio is assumed to evolve freely when either above or increasing from a specified lower consolidation threshold, e_c

$$Z_s\frac{\partial e}{\partial t} = \begin{cases} m & \text{if } e > e_c \\ m & \text{if } e = e_c \text{ and } Z_s = Z_0 \text{ and } m > 0 \\ 0 & \text{otherwise} \end{cases}, \tag{20}$$

where Z_0 is the maximum available till thickness.

When the void ratio reaches e_c from above, till begins freezing on as a frozen fringe (Rempel, 2007). Z_s, the current thickness of unfrozen till (without void space) can be modeled accordingly

$$e\frac{\partial Z_s}{\partial t} = \begin{cases} m & \text{if } e = e_c \text{ and } 0 < Z_s < Z_0 \\ m & \text{if } e = e_c \text{ and } Z_s = Z_0 \text{ and } m < 0 \\ 0 & \text{otherwise} \end{cases}. \tag{21}$$

The basal shear stress is calculated from the basal velocity and void ratio assuming that the till behaves as a Coulomb plastic material

$$\tau_b = \tau_c\frac{u_b}{\sqrt{u_b^2 + \epsilon_u^2}}, \tag{22}$$

where ϵ_u is the velocity scale over which till transitions from a quasi-linear to Coulomb friction law. The critical failure strength of the till follows the empirical form of Tulaczyk et al. (2000a)

$$\tau_c = \tau_0 \exp[-b(e - e_c)], \tag{23}$$

where τ_0 and b are empirical parameters.

1.7 Boundary Conditions

1.7.1 Flotation at the Grounding Line

Ice begins to float at the grounding line ($\sigma = 1$), so the flotation condition must apply

$$\rho_i h = \rho_w b \tag{24}$$

where, ρ_i is the density of ice, ρ_w is the density of water, h is the ice thickness, and b is the bed elevation.

Longitudinal stress is assumed to balance water pressure at the grounding line (Shumskiy and Krass, 1976)

$$\left[2\bar{A}^{-\frac{1}{n}} h \left|\frac{\partial u_b}{\partial x}\right|^{\frac{1}{n}-1} \frac{\partial u_b}{\partial x}\right]\Bigg|_{x=x_g} = \frac{1}{2}\rho_i \left(1 - \frac{\rho_i}{\rho_w}\right) g h(x_g)^2. \tag{25}$$

1.7.2 Ice Divide

By definition, the upstream boundary ($\sigma = 0$) is the ice divide. Here, we have $u_b = 0$.

1.7.3 Temperature

There are Dirichlet boundary conditions on temperature at the upper and lower ice surfaces

$$T(z = b) = T_{MP} \tag{26}$$

$$T(z = b + h) = T_s \tag{27}$$

where T_{MP} is the melting point of ice and T_s is a prescribed ice surface temperature.

At the up- and downstream boundaries, we have set zero Neumann boundary conditions

$$\frac{\partial T}{\partial x}\bigg|_{x=0} = \frac{\partial T}{\partial x}\bigg|_{x=x_g} = 0 \tag{28}$$

2 Numerics

Keep in mind that we use τ (without a subscript) in this section to indicate time. It is a placeholder in the transformed coordinate system, though we still say that $\tau = t$.

2.1 Horizontal Velocity

Following Schoof (2006), we use a variational approach to calculate horizontal velocity at the bed. The key difference here is that we have replaced the resolved lateral velocity variation with the integrated form of Dupont and Alley (2005) in our momentum balance (equation 3). We also remind the reader that τ_d is defined as in equation 4. To obtain the weak variational form, we start by multiplying the momentum balance by a test function, q

$$q\partial_x \left(2h\nu\partial_x u\right) - G_s h u^{\frac{1}{n}} q - \tau_b(u, \ldots)q - \tau_d q = 0. \tag{29}$$

Integrating over the domain (in dimensional coordinates)

$$\int_0^{x_g} \left[q\partial_x \left(2h\nu\partial_x u\right) - G_s h u^{\frac{1}{n}} q - \tau_b(u, \ldots)q - \tau_d q\right] dx = 0, \tag{30}$$

and then integrating by parts in the first term

$$\int_0^{x_g} q\partial_x \left(2h\nu\partial_x u\right) dx = \int_0^{x_g} -2h\nu\partial_x u \partial_x q dx + 2qh\nu\partial_x u|_0^{x_g}. \tag{31}$$

If the test function q satisfies zero dirichlet conditions where u has Dirichlet conditions ($u(x=0)=0$), then: $q(x=0)=0$. So

$$2qh\nu\partial_x u|_0^{x_g} = 2qh\nu\partial_x u|_{x=x_g} \tag{32}$$

where we know the RHS from our stress condition at the downstream boundary (equation 25). This term can now be written as

$$T_f q = \frac{1}{2}\rho_i\left(1-\frac{\rho_i}{\rho_w}\right)gh^2 q. \tag{33}$$

Now, substituting back into equation 30, we arrive at the weak variational form

$$\int_0^{x_g}\left[-2h\nu\partial_x u\partial_x q dx - G_s h u^{\frac{1}{n}} q - \tau_b(u,\ldots)q - \tau_d q\right]dx + T_f q = 0. \tag{34}$$

This gives a continuous functional of the form (now re-expanding the effective viscosity)

$$J(u) = -T_f u(x_g) + \int_0^{x_g}\left[\frac{2h\bar{B}}{\frac{1}{n}+1}\left|\frac{\partial u}{\partial x}\right|^{\frac{1}{n}+1} + \frac{G_s}{\frac{1}{n}+1}h|u|^{\frac{1}{n}+1} + \int_0^u \tau_b(u',\ldots)du' + \tau_d u\right]dx \tag{35}$$

We assume that u varies piecewise linearly between nodes, and calculate the integrals over nonlinear functions of u using a composite trapezoidal rule

$$J(u_i) = -T_f u_N + \sum_{i=1}^{N-1}\left[\frac{2}{\frac{1}{n}+1}\left(\frac{h_i\bar{B}_i + h_{i+1}\bar{B}_{i+1}}{2}\right)\left|\frac{u_{i+1}-u_i}{x_{i+1}-x_i}\right|^{\frac{1}{n}+1} + \right.$$
$$\left.\frac{G_s}{\frac{1}{n}+1}\left(\frac{h_i|u_i|^{\frac{1}{n}+1} + h_{i+1}|u_{i+1}|^{\frac{1}{n}+1}}{2}\right) + \int_0^u\left(\frac{\tau_b(u_i',\ldots)+\tau_b(u_{i+1}',\ldots)}{2}\right)du' + \tau_d\left(\frac{u_i+u_{i+1}}{2}\right)\right] \tag{36}$$

The resulting minimization problem is straightforward to solve using a Newton method with a Brent-type line search algorithm (Press et al., 1988)

2.2 Ice Thickness

To discretize the ice thickness ODE (equation 12), we upwind the effective velocity $u-\sigma\dot{x}_g$. We use a regularized Heavyside function to switch the direction of upwinding depending on the sign of the effective velocity. The resulting discretization looks like

$$\Delta\sigma\frac{h_i^{k+1}-h_i^k}{\Delta\tau} + \left[\frac{\bar{u}^k_{i+\frac{1}{2}} - \sigma_{i+\frac{1}{2}}(x_g^{k+1}-x_g^k)/\Delta\tau}{x_g^{k+1}}\right]\left[h_{i+1}^{k+1}\left(1-\theta^k_{i+\frac{1}{2}}\right) + h_i^{k+1}\theta^k_{i+\frac{1}{2}}\right]$$
$$-\left[\frac{\bar{u}^k_{i-\frac{1}{2}} - \sigma_{i-\frac{1}{2}}(x_g^{k+1}-x_g^k)/\Delta\tau}{x_g^{k+1}}\right]\left[h_{i-1}^{k+1}\theta^k_{i-\frac{1}{2}} + h_i^{k+1}\left(1-\theta^k_{i-\frac{1}{2}}\right)\right] +$$
$$\frac{h_i^k}{\Delta\tau}\left(1-\frac{x_g^k}{x_g^{k+1}}\right)\Delta\sigma = a\left(\sigma_i, x_g^{k+1}\right)\Delta\sigma \tag{37}$$

We evaluate θ using the previous time step to avoid non-convergence of a Newton scheme due to the large derivatives of H when the effective advection velocity $u - \sigma \dot{x}_g$ changes direction (at the expense of a few more iterations in the solver in the rare case when the effective velocity switches direction - typically during activation). θ is a regularized Heavyside function looking like

$$\theta^k_{i+\frac{1}{2}} = \frac{1}{2}\left[1 + \tanh\left[S_\theta\left(u^k_{i+\frac{1}{2}} - \sigma_{i+\frac{1}{2}}\left(\frac{x^k_g - x^{k-1}_g}{\Delta\tau}\right)\right)\right]\right] \tag{38}$$

with scale factor S_θ, which can be varied to control the shape of the function from a step function (S_θ large) to a smoother function.

In using a backward Euler scheme, we have nonlinear terms in h^{k+1} and x^{k+1}_g. Thus, we utilize a straightforward Newton scheme to solve simultaneously for ice thickness and grounding line position at each time step.

2.3 Ice Temperature

Temperature can be discretized in much the same fashion as ice thickness, though here we have not adopted a finite volume form. We use upwinding schemes in both x and z

$$\begin{aligned}
&\frac{T^{k+1}_{i,j} - T^k_{i,j}}{\Delta\tau} + \left[\frac{u^{k+1}_{i+\frac{1}{2},j} - \sigma_{i+\frac{1}{2}}(x^{k+1}_g - x^k_g)/\Delta\tau}{x^{k+1}_g\Delta\sigma}\right]\left[T^{k+1}_{i+1,j}\left(1 - \theta^k_{i+\frac{1}{2}}\right) + T^{k+1}_{i,j}\theta^k_{i+\frac{1}{2}}\right] \\
&\quad - \left[\frac{u^{k+1}_{i-\frac{1}{2},j} - \sigma_{i-\frac{1}{2}}(x^{k+1}_g - x^k_g)/\Delta\tau}{x^{k+1}_g\Delta\sigma}\right]\left[T^{k+1}_{i-1,j}\theta^k_{i-\frac{1}{2}} + T^{k+1}_{i,j}\left(1 - \theta^k_{i-\frac{1}{2}}\right)\right] + \\
&\left[w^{k+1}_{i,j+\frac{1}{2}} - \eta_{j+\frac{1}{2}}\left\{\frac{h^{k+1}_i - h^k_i}{\Delta\tau} + \left(\frac{u^{k+1}_{i,j+\frac{1}{2}} - \sigma_i(x^{k+1}_g - x^k_g)/\Delta\tau}{x^{k+1}_g}\right)\left(\frac{h^{k+1}_{i+1} - h^{k+1}_{i-1}}{2\Delta\sigma}\right)\right\}\right] \times \\
&\quad \left[T^{k+1}_{i,j+1}\left(1 - {\theta'}^k_{i,j+\frac{1}{2}}\right) + T^{k+1}_{i,j}{\theta'}^k_{i,j+\frac{1}{2}}\right] + \\
&\left[w^{k+1}_{i,j-\frac{1}{2}} - \eta_{j-\frac{1}{2}}\left\{\frac{h^{k+1}_i - h^k_i}{\Delta\tau} + \left(\frac{u^{k+1}_{i,j-\frac{1}{2}} - \sigma_i(x^{k+1}_g - x^k_g)/\Delta\tau}{x^{k+1}_g}\right)\left(\frac{h^{k+1}_{i+1} - h^{k+1}_{i-1}}{2\Delta\sigma}\right)\right\}\right] \times \\
&\quad \left[T^{k+1}_{i,j}\left(1 - {\theta'}^k_{i,j-\frac{1}{2}}\right) + T^{k+1}_{i,j-1}{\theta'}^k_{i,j-\frac{1}{2}}\right] + \frac{T^{k+1}_{i,j}}{\Delta\tau}\left(1 - \frac{x^k_g}{x^{k+1}_g}\right) + \\
&\frac{1}{h^{k+1}_i}\left[\frac{h^{k+1}_i - h^k_i}{\Delta\tau} + \left(\frac{u^{k+1}_{i,j} - \sigma_i(x^{k+1}_g - x^k_g)/\Delta\tau}{x^{k+1}_g}\right)\left(\frac{h^{k+1}_{i+1} - h^{k+1}_{i-1}}{2\Delta\sigma}\right)\right] = \frac{\kappa}{h_i}\left(\frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{\Delta\eta}\right).
\end{aligned} \tag{39}$$

Note the introduction of another Heavyside function here, θ', for which the operative variable is the vertical effective velocity (equation 18).

As this system of equations is linear in temperature, it can be solved in a straightforward fashion.

2.4 Till Water Content

The basic form of the evolution equation involves only local meltwater production without lateral transport. We use a forward Euler method to allow for enthalpy corrections to be made at grid points that are transitioning from meltwater production to till freezing and vice-versa. We begin by discretizing the equations 20 and 21

$$Z_{s,i}^k \frac{{e'}_i^{k+1} - e_i^k}{\Delta\tau} = \begin{cases} m & \text{if } e_i^k > e_c \\ m & \text{if } e_i^k = e_c \text{ and } Z_{s,i}^k = Z_0 \text{ and } m > 0 \\ 0 & \text{otherwise} \end{cases} , \tag{40}$$

$$e_i^k \frac{{Z'}_{s,i}^{k+1} - Z_{s,i}^k}{\Delta\tau} = \begin{cases} m & \text{if } e_i^k = e_c \text{ and } 0 < Z_{s,i}^k < Z_0 \\ m & \text{if } e_i^k = e_c \text{ and } Z_{s,i}^k = Z_0 \text{ and } m < 0 \\ 0 & \text{otherwise} \end{cases} . \tag{41}$$

Here we have introduced a prime notation on void ratio and till thickness that indicates these variables have not yet been corrected for crossing over thresholds. We do not want to overshoot and make $e < e_c$ or $Z_s > Z_0$. The corrected cases are based on the idea that we would like to take the change past a threshold in one variable and translate to a change from a threshold in another variable. To do so, we start with the following equality

$$Z_0 \frac{e' - e_c}{\Delta\tau} = e_c \frac{Z'_s - Z_0}{\Delta\tau} \tag{42}$$

If $Z'_s > Z_0$, then we need to translate this extra change to a change in e', which we can solve for

$$e' = e_c \left(\frac{Z'_s}{Z_0} \right) . \tag{43}$$

If $e' < e_c$, then we need to translate this extra change to a change in Z'_s, which we can solve for

$$Z'_s = Z_0 \left(\frac{e'}{e_c} \right) \tag{44}$$

We can write these into case-by-case corrections

$$e_i^{k+1} = \begin{cases} e_c & \text{if } {e'}_i^{k+1} < e_c \\ e_c \left(\frac{{Z'}_{s,i}^{k+1}}{Z_0} \right) & \text{if } {Z'}_{s,i}^{k+1} > Z_0 \\ {e'}_i^{k+1} & \text{otherwise} \end{cases} , \tag{45}$$

$$Z_{s,i}^{k+1} = \begin{cases} Z_0 \left(\frac{{e'}_i^{k+1}}{e_c} \right) & \text{if } {e'}_i^{k+1} < e_c \\ Z_0 & \text{if } {Z'}_{s,i}^{k+1} > Z_0 \\ {Z'}_{s,i}^{k+1} & \text{otherwise} \end{cases} . \tag{46}$$

2.5 Discretization

In this model, the grid is staggered in the horizontal. Most variables which are only defined in the x direction (h, e, Z_s) are located on the elements. Horizontal velocity u, is defined on grid box corners. T is defined on grid box centers. w is defined on grid box edges.

Thickness at the grounding line is used in order to maintain the flotation condition at the outer boundary. However, because thickness is defined on elements in our discretization scheme, we must add an additional equation to solve for thickness and grounding line position simultaneously (as detailed in section 2.2. Ice thickness in the two grid points near the grounding line is relaxed to the grounding line thickness (which is determined through the flotation condition)

$$h_{GL} = \frac{3}{2}h_{N-\frac{1}{2}} - \frac{1}{2}h_{N-\frac{3}{2}}. \tag{47}$$

This approach is also described in the appendix of Schoof (2007).

2.6 Horizontal Grid Refinement

The approach we have taken here allows us to define an arbitrary mesh in $\sigma - \eta$ space. For simplicity, we use equally-spaced η coordinates.

As Schoof (2007) has shown, in order to accurately simulate transient grounding line migration, we must resolve the mechanical grounding zone transition from ice sheet to ice shelf flow. The σ coordinates are correspondingly refined in a fashion similar to Schoof (2007), by defining a grounding zone in sigma (here we use $\sigma \in [0.97\ \ 1]$) where the resolution is high (~100 m). In most of the ice stream not in the grounding zone a lower resolution is used. We explore the convergence of solutions in increasing upstream resolution in section 3.3.

3 Comparison to Robel et al. 2013 (Figure 1)

In order to make a valid comparison between the flowline model described here and that described in Robel et al. (2013), we need to be able to map parameters between both models. In the case of most parameters (see table of parameters in both studies), this is fairly straightforward. For some parameters there is no direct translation between models.

In Robel et al. (2013), there is an ice stream length, L, which does not translate to this model, which has a migrating grounding line. Here we will assume that ice stream length is the grounding line position in the steady-streaming regime, before oscillations are induced $L = 770$ km. This remains within about 15% of the grounding line position during oscillations.

In Robel et al. (2013), there is an ice stream width, W, which is not explicitly set in this model. However, we do use an ice stream half-width of $W = 25$ km to set the lateral shear stress parameter, $G_s = 400$, and so we use this as a comparison to Robel et al. (2013). Additionally, we note that the Glen's flow law coefficient, A_g that shows up in both Robel et al. (2013) and G_s are both taken in the shear margins, so we set them to $A_g = 2.7 \times 10^{-24}$ Pa^{-3} s^{-1}, appropriate for temperate shear margins.

The one parameter (with dynamical significance) that does not have an equivalent between these two studies is bed slope, b_x. Robel et al. (2013) assumes that the ice stream rests on a flat bed. Setting a flat bed in this study would lead the implicit thickness solver to find non-unique solutions for the grounding line position. As such we must have some non-zero bed slope. This study only includes simulations with bedslope $b_x = 5 \times 10^{-4}$. In order to adapt the analytic prediction of stability boundary location from Robel et al. (2013) (refer to supplementary material), we can add a bed slope term, α, into the driving stress scaling

$$[\tau_d] = \rho_i g[h] \left(\frac{[h]}{L} + \alpha \right). \tag{48}$$

We use the same steady-state balance between accumulation and mass loss from ice streaming: $a_c L = [h][u_b]$, which leads to an equation for the thickness scale

$$a_c L = \frac{A_g W^{n+1}[h]}{4^n(n+1)} \left[\frac{\rho_i g[h]}{L} \left(1 + \frac{\alpha L}{[h]} \right) \right]^n. \tag{49}$$

This does not permit an exact closed-form solution for $[h]$ as before, so we solve numerically for $[h]$ for the purposes of this comparison.

Though we have numerically solved for exact $[h]$ to produce Figure 1 of this study, there are perturbation methods for developing closed-form approximations for the case when α is small. We start with the following form for the approximation (dropping brackets)

$$h(\alpha) \approx h_0 + h_1 \alpha + h_2 \alpha^2. \tag{50}$$

We plug this into equation 49 and start by setting $\alpha = 0$, and deriving the same zero-slope approximation that is in equation 12 of the supplementary material of Robel et al. (2013)

$$h_0 = L \left[\frac{A_g W^{n+1} (\rho_i g)^n}{4^n(n+1)a_c} \right]^{-\frac{1}{n+1}}. \tag{51}$$

We then expand equation 49, canceling the terms corresponding to the $\alpha = 0$ approximation, and then retaining terms in α^1

$$0 = Q \left[4h_0^3 h_1 \alpha + 3\alpha L h_0^3 \right]. \tag{52}$$

Solving for h_1 gives

$$h_1 = -\frac{3L}{4}. \tag{53}$$

Moving onto the second order approximation, we return to the expanded version of 49, retaining only terms in α^2

$$0 = Q \left[4h_0^3 h_2 \alpha^2 + 6h_0^2 h_1^2 \alpha^2 + 3\alpha L \left(3h_0^2 h_1 \alpha \right) + 3\alpha^2 L^2 h_0^2 \right]. \tag{54}$$

Solving for h_2

$$h_2 = \frac{3}{32} \frac{L^2}{h_0}. \tag{55}$$

These give the following second-order approximation for h

$$h \approx h_0 - \frac{3L}{4}\alpha + \frac{3}{32} \frac{L^2}{h_0} \alpha^2, \tag{56}$$

which gives an approximation that is less than 1% from the exact solution for the parameters used in this study. We can then plug this into our approximation for the stability boundary location.

References

Dupont, T. and Alley, R. (2005). Assessment of the importance of ice-shelf buttressing to ice-sheet flow. *Geophys. Res. Lett.*, 32(L04503).

Press, W., Teukolsky, S., Vetterline, W., and Flannery, B. (1988). *Numerical Recipes in C.* Cambridge university press.

Rempel, A. (2007). A theory for ice-till interaction and sediment entrainment beneath glaciers. *J. Geophys. Res.*, 113:F01009, 1–17, doi:10.1029/2010JF001839.

Robel, A., DeGiuli, E., Schoof, C., and Tziperman, E. (2013). Dynamics of ice stream temporal variability: Modes, scales and hysteresis. *J. Geophys. Res.*, 118:925–936.

Schoof, C. (2006). A variational approach to ice stream flow. *J. Fluid Mech.*, 556:227–251.

Schoof, C. (2007). Ice sheet grounding line dynamics: Steady states, stability, and hysteresis. *Journal of Geophysical Research - Earth Surface*, 112(F3).

Shumskiy, P. and Krass, M. (1976). Mathematical models of ice shelves. *J. Glaciol.*, 17(77):419–432.

Tulaczyk, S., Kamb, W., and Engelhardt, H. (2000a). Basal mechanics of Ice Stream B, West Antarctica 1. Till mechanics. *Journal of Geophysical Research-Solid Earth*, 105(B1):463–481.

Tulaczyk, S., Kamb, W., and Engelhardt, H. (2000b). Basal mechanics of Ice Stream B, West Antarctica 2. Undrained plastic bed model. *Journal of Geophysical Research-Solid Earth*, 105(B1):483–494.